Discrete Applied Mathematics 157 (2009) 1069–1085

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Extensive facility location problems on networks with equity measures

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ARTICLE INFO

Article history:
Received 20 July 2007
Accepted 28 March 2008
Available online 19 May 2008

Keywords:
Path location
Range criterion
Hurwicz criterion
Pareto-optimal paths

ABSTRACT

This paper deals with the problem of locating path-shaped facilities of unrestricted length on networks. We consider as objective functions measures conceptually related to the variability of the distribution of the distances from the demand points to a facility. We study the following problems: locating a path which minimizes the range, that is, the difference between the maximum and the minimum distance from the vertices of the network to a facility, and locating a path which minimizes a convex combination of the maximum and the minimum distance from the vertices of the network to a facility, also known in decision theory as the Hurwicz criterion. We show that these problems are NP-hard on general networks. For the discrete versions of these problems on trees, we provide a linear time algorithm for each objective function, and we show how our analysis can be extended also to the continuous case.

1. Introduction

Network facility location problems consist of locating a specified number of facilities in a network in order to supply a set of costumers. Commonly used objective functions are either the sum of the distances from each client to its nearest facility (*median* criterion), or the maximum of these distances (*center* criterion). Starting from the location of one or a set of points, which can be either vertices or points along the edges, several authors extended the theory to facilities with a connected structure (*extensive* facilities), such as path-shaped or tree-shaped facilities [5,17,19]. For a comprehensive review, see, for example, [3,15,21]. Although median and center are the most representative objective functions in location problems, none of these two criteria alone is able to capture all the essential elements of a location problem. In recent years some papers considered the problem of finding an optimal location of a path or a tree using the two criteria simultaneously, or a convex combination of them [1,2,13,20], or by considering the general ordered median objective [16]. Nevertheless, also in these cases some salient features of real problems, like the dispersion of the clients' demand with respect to a facility, are not captured. The dispersion is a concept strictly related to the variability of the distribution of the distances from the demand points to a facility. In particular, the issue of equity seems to be relevant when locating facilities in the public sector applications. In the literature there are two main lines of research about equity measures. The first one deals with how to measure equity and which properties equity measures should have. The second line of research is concerned with providing efficient algorithms for the location of facilities in a network according to some equity measure. A review of the existing literature about equity measures in location theory is given in [10]. In point location problems, efficient algorithms have been provided for the minimization of the variance of the distance travelled by a customer to a facility, as well as, for minimizing the range objective function, which is given by the difference between the maximum and the minimum

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2008.03.035

distance from a facility [11]. Almost all the papers focusing on equity measures deal with the location of a single point on a network. An exception is [4] where the authors provide an $O(n^2 \log n)$ time algorithm for the location, on a tree network, of a path-shaped facility which minimizes the variance. In [14] the same problem is solved in $O(n^2)$ time.

In this paper we consider the range objective function, as well as, the Hurwicz objective, which originates in decision theory [7,12] and is given by a convex combination of the maximum and the minimum distance from the vertices of the network to the facility. We study the problem of locating path-shaped facilities of unrestricted length on a tree where all the vertices have the same weight, while positive real lengths are associated to the edges. We focus on two main problems: locating a path which minimizes the range, and locating a path which minimizes the Hurwicz objective function. Moreover, we study the following two additional range-type constrained optimization problems: locating a path which minimizes the maximum distance subject to the minimum distance bounded below by a constant, and locating a path which maximizes the minimum distance subject to the maximum distance bounded above by a constant. Similarly, we study the following two additional Hurwicz-type problems: locating a path which minimizes the maximum distance subject to the minimum distance bounded above by a constant, and locating a path which minimizes the minimum distance subject to the maximum distance bounded above by a constant. To the best of our knowledge these six problems have not been considered in the literature, yet. We consider both the discrete version of the above problems, that is, when the endpoints of the path are vertices, and the continuous version, that is, when at least one endpoint belongs to the interior of an edge. For general networks we show that both versions of the six problems are NP-hard. For tree networks we provide the following results. In the discrete case, we solve all the six problems by adopting a bicriterion approach similar to those provided in [1,15], that is, we embed each of them into a suitable bicriteria problem related to the maximum and the minimum distance criteria. We provide two linear time dynamic programming algorithms for the range-type and Hurwicz-type problems, respectively. Given a tree with n vertices, each algorithm finds in $O(n)$ time a superset that includes the representation of the set of Pareto-optimal paths in the outcome space, along with some extra points. We also show that the cardinality of this superset is $O(n)$. Thus, solving the discrete versions of the above six problems is done in linear time by scanning this superset and evaluating the objective functions at each of its elements. Moreover, we show that the set of Pareto-optimal paths can be extracted from this superset in $O(n \log n)$ time by using the algorithm provided by Kapoor [8]. In the continuous case, we provide an $O(n^2)$ time algorithm to solve the problem of minimizing the range function. We also provide algorithms that solve the range-type constrained optimization problems in $O(n)$ time. For the Hurwicz-type problems we show that either no optimal path exists, or it reduces to the problem of locating a path which minimizes the maximum distance.

For the implementation of our algorithms we need some quantities associated to each vertex of the tree. These quantities are computed in a preprocessing phase in time $O(n)$. Some of the recursive functions adopted in this phase are already known in the literature, but some others are presented in this paper for the first time.

The paper is organized as follows. In Section 2, we introduce the notation and definitions and prove the complexity results on general graphs. Section 3 provides the recursive formulas for computing all the quantities needed in our algorithms. Section 4 describes the algorithms for solving the range-type and the Hurwicz-type problems on trees. The paper ends with some concluding remarks and extensions.

2. Definitions, complexity results and basic properties

Let $T = (V, E)$ be a tree with $|V| = n$. Suppose that a weight equal to one is associated to each vertex of the tree, while a positive real length $\ell(e)$ is assigned to each edge $e \in E$. Suppose that T is rooted at a vertex r and denote by T_r the rooted tree. For any vertex v, let T_v be the subtree of T_r rooted at vertex v, $S(v)$ the set of children of v in T_r, and $p(v)$ the parent of v in T_r. Clearly, a vertex v is a leaf if and only if $|S(v)| = 0$. For any pair of points x and y in T, that may be vertices or may belong to the interior of an edge, we denote by $P(x, y)$ the unique path connecting x and y. We denote by $d(x, y)$ the length of $P(x, y)$. In the following, we will avoid specifying one or both the endpoints of a path if not necessary. A path is *discrete* if both its endpoints are vertices of T, otherwise it is *continuous*. We denote by *diam* the diameter of T, i.e., the length of a longest path in T, and by c its absolute center, i.e., the middle point of the longest path in T. We denote by $d(u, P)$ the distance from a vertex u to a path P, that is, the length of the shortest path from u to a vertex or an endpoint of P. For any point x in T, the eccentricity of x is $E(x) = \max_{u \in V} d(u, x)$, while for any path P the eccentricity of P is $E(P) = \max_{u \in V} d(u, P)$. The absolute center c is the point in T that minimizes the eccentricity. The absolute center could be either a vertex, or a point along an edge. Finally, PC denotes the path center of T, that is, the shortest path that minimizes the eccentricity in T. It is well known that $c \in PC$ and that PC is unique [6,18].

For a tree $T = (V, E)$, we consider the range objective function which is defined as follows:

$$R(P) = \max_{u \in V \setminus P} d(u, P) - \min_{u \in V \setminus P} d(u, P). \tag{1}$$

Given a path P, for any $0 \leq \alpha \leq 1$, we also consider the Hurwicz objective function:

$$H(P) = \alpha \max_{u \in V \setminus P} d(u, P) + (1 - \alpha) \min_{u \in V \setminus P} d(u, P). \tag{2}$$

Both $R(P)$ and $H(P)$ are non negative variability measures. In this paper we suppose that the tree T is not a path. Actually, since we are concerned with *desirable* facilities, when T is a path, the solution is assumed to be the path itself.

Table 1
Summary of results

Range-type problems				Hurwicz-type Problems			
Problem		Discr.	Cont.	Problem		Discr.	Cont.
P1	$\min R(P)$	$O(n)$	$O(n^2)$	P4	$\min H(P)$	$O(n)$	No optimal path exists
P2	$\min\ E(P) s.t. \mu(P) \geq \gamma$	$O(n)$	$O(n)$	P5	$\min\ E(P) s.t. \mu(P) \leq \gamma$	$O(n)$	$O(n)$
P3	$\max \mu(P) s.t.\ E(P) \leq \gamma$	$O(n)$	$O(n)$	P6	$\min\ \mu(P) s.t.\ E(P) \leq \gamma$	$O(n)$	No optimal path exists

Given a path P we denote by $\mu(P) = \min_{u\in V\setminus P} d(u, P)$ the minimum distance from a vertex $u \notin P$ to P. Given a subset $I \subset V$ and a path P whose vertices are all in I, we denote by $\mu_I(P) = \min_{u\in I\setminus P} d(u, P)$ the minimum distance to P from any vertex $u \in I$ not belonging to P. Clearly, $\mu_V(P) = \mu(P)$.

Given a path P, since $d(u, P) = 0$ for each $u \in P$, (1) and (2) can be rewritten in the equivalent form:

$$R(P) = E(P) - \mu(P) \tag{3}$$

and

$$H(P) = \alpha E(P) + (1 - \alpha)\mu(P). \tag{4}$$

In this paper we study six different path location problems of unrestricted length, both in their discrete and continuous version (see, Table 1). We show that these problems are NP-hard on general networks, while we provide new complexity results for all the problems on trees. These results are summarized above.

Problems P1 and P4 are unrestricted optimization problems, while P2, P3, P5, and P6 arise when we want to locate a path which optimizes one criterion subject to a constraint on the other. Note that, problem P2 does not have any solution if $\gamma > \max\{\ell(e)|e \in E\}$, while the discrete version of problem P5 does not have any solution if $\gamma < \min\{\ell(e)|e \in E\}$. For problems P3 and P6 we assume $E(PC) \leq \gamma$, otherwise the problems are infeasible.

To the best of our knowledge the continuous and discrete versions of problems P1–P6 have not been considered in the literature yet, either on general networks, or on trees.

It can be shown that problems P1–P6 are NP-hard on general networks, both in the continuous and in the discrete version. We first consider the discrete case.

Problems P2 and P5 contain as a special case the problem of finding a path that minimizes the maximum distance from the vertices of a network to the facility. For P2 this happens when $\gamma \leq \min\{\ell(e)|e \in E\}$, while for P5 this happens when $\gamma \geq \max\{\ell(e)|e \in E\}$. Problem P4 contains as a special case the problem of finding a path that minimizes the maximum distance when $\alpha = 1$. Thus, Problems P2, P4, and P5 are NP-hard on general networks [5].

It can be shown that also problems P1, P3 and P6 are NP-hard on general networks by using arguments similar to those given in [5]. Let us start with the decision version of problem P1. Given an arbitrary graph $G = (V, E)$ and a non negative number R_0, decide if there exists a path P such that $R(P) \leq R_0$. We show that the Hamiltonian Path problem can be reduced to this problem. Let $|V| = n$, and suppose that a length equal to one is assigned to each edge $e \in E$. For each $v_i \in V$, $i = 1, \ldots, n$, consider two additional vertices v_{i1} and v_{i2}, and construct a new graph $G' = (V', E')$ such that $V' = V \cup \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}\}$ and $E' = E \cup \bigcup_{i=1}^{n}\{(v_i, v_{i1}), (v_i, v_{i2})\}$. Assume that the edges added to G have length equal to 1/2. Set $R_0 = 0$. It is easy to see that problem P1 has a solution in G' if and only if G has a Hamiltonian Path.

Let us now consider the decision version of problem P3: given two non negative numbers μ_0 and M, decide if there exists a path P such that $\mu(P) \geq \mu_0$ and $E(P) \leq M$. We refer to the same reduction as above. We assign length equal to 1 to the original edges $e \in E$, and length equal to M to the new edges. It is easy to see that, by setting $\mu_0 = 2$ and $M \geq 2$, problem P3 has a solution in G' if and only if G has a Hamiltonian Path.

Finally, consider problem P6, and the same reduction as above. We still assign length equal to 1 to the original edges $e \in E$, and length equal to 1/2 to the new edges (v_i, v_{i1}), $i = 1, \ldots, n$, while a length equal to $M \gg 0$ is assigned to the new edges (v_i, v_{i2}), $i = 1, \ldots, n$. Consider the decision version of problem P6: given a positive number μ_0, decide if there exists a path P such that $\mu(P) \leq \mu_0$ and $E(P) \leq M$. Set $\mu_0 = 1/2$. Also in this case, it is easy to see that problem P6 has a solution in G' if and only if G has a Hamiltonian Path.

For the continuous versions of problems P2, P4 and P5, NP-hardness follows since each of them contains as a special case the problem of finding a continuous path that minimizes the maximum distance from the vertices of a network to the facility [5]. For the continuous versions of problems P1, P3 and P6, NP-hardness can be proved applying the same reduction as in the discrete case.

Now we turn to consider the discrete version of our problems on trees. In the following we study the case when the absolute center is a vertex of T. Later, we will show that our analysis applies also when the absolute center is a point along an edge of T.

Let Π denote the set of all discrete paths of the tree T. For solving problems P1–P3, it is necessary to find the maximum of the minimum distances to a path $P \in \Pi$, while for solving P4–P6 we have to find the minimum of the minimum distances to a path $P \in \Pi$. Given a set $S \subseteq \Pi$, we define $M_S = \max_{P\in S} \mu(P)$, for problems P1–P3, and $m_S = \min_{P\in S} \mu(P)$, for problems P4–P6.

We root T at the absolute center c. Suppose that $PC \neq \{c\}$, then we partition the set of vertices of T_c in the following way. Denote by v_1 one of the two vertices adjacent to c along PC. Let $T_1 = T_{v_1} = (V_1, E_1)$ be the subtree of T_c rooted at v_1, and let

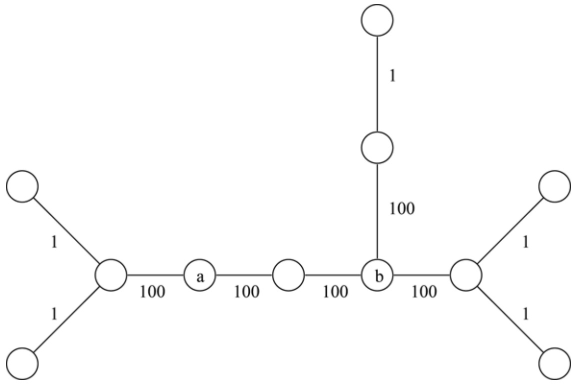

Fig. 1. The path $P = P(a, b)$ has $E(P) = 101$ and $\mu(P) = 100$. Any path connecting two leaves is dominated by P w.r.t. $\succeq_1$.

$T_2 = (V_2, E_2)$, with $V_2 = V \setminus V_1$. Clearly, $c \in V_2$. Thus, once T_2 has been identified, we consider T_2 rooted at c. On the basis of this decomposition, Π can be partitioned into the three following sets:

- $\mathcal{P}^1$, the paths that contain only vertices of V_1;
- $\mathcal{P}^2$, the paths that contain only vertices of V_2;
- $\overline{\mathcal{P}}$, the paths that pass through c and have exactly one endpoint in V_1.

Note that, if $PC \neq \{c\}$, any path $P \in \overline{P}$ intersects PC in at least one edge.

When $PC = \{c\}$ we do not need to partition either the tree, or the set Π. Actually, in this case the rooted tree T_c is analyzed as a whole.

For problems P1–P6, the idea of all the algorithms is similar to those presented in [1,15]. We consider our six problems embedded into bicriteria path problems with respect to the functions $E(\cdot)$ and $\mu(\cdot)$. More precisely, we define the following two partial orders. Given a path P, a point $(E(P), \mu(P))$ is non-dominated in the partial order $\succeq_1$, i.e., is a *Pareto-optimal point with respect to* $\succeq_1$, if there is no other path P^* with $(E(P^*), \mu(P^*))$ such that $E(P^*) \le E(P)$, $\mu(P^*) \ge \mu(P)$, and $E(P^*) - \mu(P^*) < E(P) - \mu(P)$. A point $(E(P), \mu(P))$ is non-dominated in the partial order $\succeq_2$, i.e., is a *Pareto-optimal point with respect to* $\succeq_2$, if there is no other path P^* with $(E(P^*), \mu(P^*))$ such that $E(P^*) \le E(P)$, $\mu(P^*) \le \mu(P)$, and $E(P^*) + \mu(P^*) < E(P) + \mu(P)$. For solving problems P1–P3 and P4–P6 we are interested in finding all the Pareto-optimal paths w.r.t the partial order $\succeq_1$ and the partial order $\succeq_2$, respectively. We denote by $\pi_i \subseteq \Pi$ the set of Pareto-optimal paths with respect to the partial order $\succeq_i$, $i = 1, 2$. Note that $|\pi_i| \le O(n^2)$, since the total number of paths in T is $O(n^2)$.

Proposition 1. *Every optimal solution of problem* P1 *is also Pareto-optimal with respect to* $E(\cdot)$ *and* $\mu(\cdot)$*, for the partial order* $\succeq_1$*. Moreover, for both problems* P2 *and* P3 *there is at least an optimal solution which is also Pareto-optimal for the partial order* $\succeq_1$*. In addition, every optimal solution of problem* P4 *is also Pareto-optimal with respect to* $E(\cdot)$ *and* $\mu(\cdot)$ *for the partial order* $\succeq_2$ *and for both problems* P5 *and* P6 *there is at least an optimal solution which is also Pareto-optimal for the partial order* $\succeq_2$*.* □

From an algorithmic viewpoint, for each of the above two partial orders, we consider a superset W_i, $i = 1, 2$ that includes the set of the Pareto-optimal paths in the outcome space $(E(\cdot), \mu(\cdot))$ along with some extra paths, and we generate the following representation sets $\phi(W_i)$, $i = 1, 2$:

$$\phi(W_i) = \{(E(P), \mu(P)) \subset \mathbb{R}^2 | P \in W_i\} \quad i = 1, 2. \tag{5}$$

Given the tree rooted at c, we identify groups of paths in T_c with the same value of the eccentricity (see Section 3). For problems P1–P3, among all the paths with the same eccentricity, we search for a path P that maximizes $\mu(\cdot)$ and we include its representation in $\phi(W_1)$. Similarly, for problems P4–P6, among all the paths with the same eccentricity, we search for a path P that minimizes $\mu(\cdot)$ and we include its representation in $\phi(W_2)$.

Note that a Pareto-optimal path – both for $\succeq_1$ and $\succeq_2$ – does not necessarily connect two leaves of the tree. An example for $\succeq_1$ is shown in Fig. 1.

3. Recursive formulas

In order to solve efficiently the problems presented in this paper, a preprocessing phase is needed to compute some quantities that will be used in the algorithms. In the following we describe the recursive formulas computed during this preprocessing phase.

First of all we consider paths in $\mathcal{P}^1$ or in $\mathcal{P}^2$. We identify each path P with the vertex at which the maximum distance to P is attained from a vertex $u \notin P$. Given the rooted tree T_c and a vertex $v \in V$, denote by P_v a path in T_v passing through v or having v as an endvertex, and by $\mathcal{P}(T_v)$ the set of all such paths. Since each vertex $v \in V$ identifies all the paths in $\mathcal{P}(T_v)$, and these paths have the same eccentricity, we denote this common value by $E_{\mathcal{P}(T_v)}$. For each $v \in V$, the value of this maximum distance can be computed in constant time applying the following result provided in [1]:

Theorem 1 ([1]). *For any path $P \in \mathcal{P}(T_v)$, we have:*

$$E(P) = E_{\mathcal{P}(T_v)} = d(v, c) + \frac{\text{diam}}{2}. \quad \square \tag{6}$$

On the basis of the results of Proposition 1, our algorithms visit the rooted tree bottom-up and compute the set $\phi(\cdot)$ w.r.t. the partial orders $\succeq_1$ and $\succeq_2$.

Actually, w.r.t. paths in $\mathcal{P}^1$ and $\mathcal{P}^2$, when $\succeq_1$ is considered, for each vertex $v \in V$, among all the paths $P_v \in \mathcal{P}(T_v)$, with $E(P_v) = E_{\mathcal{P}(T_v)}$, the algorithm computes the *maximum* value of $\mu(\cdot)$ (problems P1–P3). When $\succeq_2$ is considered, for each vertex $v \in V$, among all the paths $P_v \in \mathcal{P}(T_v)$, with $E(P_v) = E_{\mathcal{P}(T_v)}$, the algorithm computes the *minimum* value of $\mu(\cdot)$ (problems P4–P6).

First of all, we consider problems P1–P3 in which, besides minimizing $E(\cdot)$, we are interested in maximizing $\mu(\cdot)$. To this purpose, among all the paths $P_v \in \mathcal{P}(T_v)$, we want to find a path which maximizes the minimum distance from a vertex $u \in T_v \setminus P_v$ to P_v. A path with such a property, will be called *best path of type 1*.

Let us define:

$$\beta(v) = \max_{\substack{P_v \in \mathcal{P}(T_v) \\ v \text{ endvertex of} P_v}} \mu_{T_v}(P_v).$$

Let

$$d_1(v) = \min_{w \in S(v)} \ell(v, w), \quad \text{with } w_1 \in \arg\min\{\ell(v, w) | w \in S(v)\};$$

$$d_2(v) = \min_{w \in S(v) | w \neq w_1} \ell(v, w), \quad \text{with } w_2 \in \arg\min\{\ell(v, w) | w \in S(v), w \neq w_1\};$$

$$d_3(v) = \min_{w \in S(v) | w \notin \{w_1, w_2\}} \ell(v, w).$$

We set $d_1(v) = +\infty$ when v is a leaf of T_c, $d_2(v) = +\infty$ when $|S(v)| \leq 1$, and $d_3(v) = +\infty$ when $|S(v)| \leq 2$.

We define the label flag(v) which is equal to 0 either if the subtree T_v is a path, or if v is a leaf, and it is equal to 1 otherwise. It can be computed in a bottom-up level-by-level visit of the rooted tree T_c. For each vertex v of T_c we have:

$$\text{flag}(v) = \begin{cases} 0 & \text{if } |S(v)| = 1 \text{ and flag}(w) = 0, \text{ or if } |S(v)| = 0 \\ 1 & \text{otherwise,} \end{cases} \tag{7}$$

where $w \in S(v)$.

Property 1. *Let $P_v \in \mathcal{P}(T_v)$ be a path starting at v and connecting v to the descendants of a child $w \neq w_1$ of v. We have:*

$$\mu_{T_v}(P_v) \leq \mu_{T_v}(\{v\}) = d_1(v). \quad \square$$

After Property 1, in order to compute $\beta(v)$, besides the single vertex v with $\mu_{T_v}(\{v\}) = d_1(v)$, we need to evaluate only the paths connecting v and the descendants of w_1, while all the other paths in T_v with v as an endvertex can be ignored. Thus, for each vertex v of T_c, $\beta(v)$ can be recursively computed as follows:

$$\beta(v) = \begin{cases} +\infty & \text{if } |S(v)| = 0 \\ d_1(v) & \text{if flag}(v) = 0 \text{ and } |S(w_1)| = 0 \\ \max\{d_1(v), \beta(w_1)\} & \text{if } |S(v)| = 1 \\ d_2(v) & \text{if } |S(v)| > 1 \text{ and flag}(w_1) = 0 \\ \max\{d_1(v), \min\{d_2(v), \beta(w_1)\}\} & \text{if } |S(v)| > 1 \text{ and flag}(w_1) = 1. \end{cases} \tag{8}$$

Note that in the computation of $\beta(v)$ we consider both the case in which the best path of type 1, $P_v \in \mathcal{P}(T_v)$ having v as an endvertex, is v itself and the case in which it includes at least one edge.

Property 2. *Let $P_v^{f,g} \in \mathcal{P}(T_v)$ be a path passing through v, with $|S(v)| \geq 3$, and connecting the descendants of two children f and g of v. Then, we have*

- *if $f = w_1$ and $g \neq w_2$, then any best path P_v starting at v and connecting v with the descendants of w_1 has $\mu_{T_v}(P_v) = d_2(v) \geq \mu_{T_v}(P_v^{w_1,g})$ (similarly if $f \neq w_2$ and $g = w_1$);*
- *if $f = w_2$ and $g \neq w_1$, then any best path P_v starting at v and connecting v with the descendants of w_2 has $\mu_{T_v}(P_v) = d_1(v) \geq \mu_{T_v}(P_v^{w_2,g})$ (similarly if $f \neq w_1$ and $g = w_2$);*
- *if $f, g \neq w_1, w_2$, then $\mu_{T_v}(\{v\}) = d_1(v) \geq \mu_{T_v}(P_v^{f,g})$.* $\square$

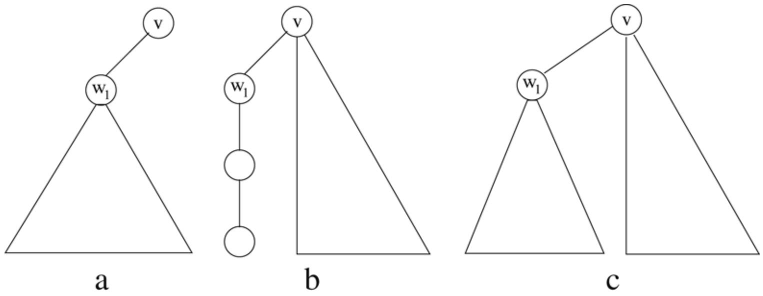

Fig. 2. Possible configurations for T_v.

After Property 2, we need to evaluate only the paths connecting v and the descendants of w_1 and w_2, while all the other paths in T_v passing through v, can be ignored. Note that for any path $P_v^{w_1,w_2}$ we always have $\mu_{T_v}(P_v^{w_1,w_2}) \leq d_3(v)$. Thus, the maximum of the minimum distances $M_{\mathcal{P}(T_v)}$ of a best path of type 1, $P_v \in \mathcal{P}(T_v)$ (passing through v or having v as an endvertex) with respect to the whole tree T_c can be computed as follows

$$M_{\mathcal{P}(T_v)} = \begin{cases} \min\{\ell(v, p(v)), \beta(v)\} & \text{if } |S(v)| \leq 1 \\ \min\{\ell(v, p(v)), \max\{\beta(v), \min\{\beta(w_1), \beta(w_2)\}\}\} & \text{if } |S(v)| = 2 \\ \min\{\ell(v, p(v)), \max\{\beta(v), \min\{\beta(w_1), \beta(w_2), d_3(v)\}\}\} & \text{if } |S(v)| \geq 3, \end{cases} \tag{9}$$

where we set $\ell(v, p(v)) = +\infty$ if v is the root of the tree.

We note that for any given rooted tree T_r, formulas (7)–(9) can be applied to T_r for finding the maximum of the minimum distances of a best path of type 1 in $\mathcal{P}(T_v)$ w.r.t. the whole tree (see the example in Section 3.1). In our approach, when $PC = \{c\}$ we must apply these formulas to T_c. On the other hand, when $PC \neq \{c\}$, formulas (7)–(9) must be computed on T_1 and T_2 separately. In this case, since the vertex c is evaluated as the root of T_2, with $E_{\mathcal{P}(T_c)} = \frac{\text{diam}}{2}$, in formula (9) we set $\ell(c, p(c)) = \ell(c, v_1)$.

Proposition 2. *The labels (8) and (9) correctly compute the maximum of the minimum distances of a best path of type 1 $P_v \in \mathcal{P}(T_v)$ from all the other vertices in T_c not belonging to P_v.*

Proof. First consider formula (8), which refers to the best paths of type 1 having v as an endvertex. When v has exactly one child corresponding to a leaf, the best path is $P_v = \{v\}$ with $\mu_{T_v}(\{v\}) = d_1(v)$. In all the other cases we base our analysis on Property 1.

If $|S(v)| = 1$ (see Fig. 2(a)), we have to compare $P_v = \{v\}$ only with a best path connecting v with the descendants of its child w_1.

When $|S(v)| > 1$ and $\text{flag}(w_1) = 0$ (see Fig. 2(b)), the maximum of the minimum distances of a best path of type 1 P_v is always equal to $d_2(v)$. In fact, consider $\beta(w_1)$: the maximum of the minimum distances to P_v will be equal to $d_2(v)$ either if $d_2(v) \leq \beta(w_1)$, or $d_2(v) > \beta(w_1)$, since, in the latter case, one can always extend P_v up to the leaf in T_{w_1}. Note that in this case the path $P_v = \{v\}$ is discarded in any case since $d_1(v) \leq d_2(v)$.

When $|S(v)| > 1$ and $\text{flag}(w_1) = 1$ (see Fig. 2(c)), if $d_2(v) \leq \beta(w_1)$, then the maximum of the minimum distances is $d_2(v)$ itself, but it is equal to $\max\{d_1(v), \beta(w_1)\}$ if $d_2(v) > \beta(w_1)$ since, in this case, also the path $P_v = \{v\}$ must be evaluated.

On the basis of Properties 1 and 2, formula (9) first identifies the best path of type 1, $P_v \in \mathcal{P}(T_v)$, comparing, when necessary, the best path having v as an endvertex with the best path passing through v and connecting v with the descendants of w_1 and w_2. Then, the maximum of the minimum distances of a best path of type 1 $P_v \in \mathcal{P}(T_v)$ w.r.t. the whole tree T_c is computed by considering also the length of the edge $(v, p(v))$. □

In all the problems P1–P3 some additional considerations arise when $PC \neq \{c\}$ and we have to compute the labels with respect to the paths in $\overline{\mathcal{P}}$.

Given a pair of vertices $p_1, p_2 \in PC$, the eccentricity of the subpath $P(p_1, p_2) \subseteq PC$ is given by

$$E(P(p_1, p_2)) = \max\left\{\frac{\text{diam}}{2} - d(p_1, c), \frac{\text{diam}}{2} - d(p_2, c)\right\}. \tag{10}$$

For any given subpath $P(p_1, p_2) \subseteq PC$, when $\frac{\text{diam}}{2} - d(p_1, c) \geq \frac{\text{diam}}{2} - d(p_2, c)$, we denote by $\mathcal{P}(p_1)$ the set of all paths in $\overline{\mathcal{P}}$ containing $P(p_1, p_2)$, and having eccentricity equal to $\frac{\text{diam}}{2} - d(p_1, c)$. A generic path belonging to $\mathcal{P}(p_1)$ will be denoted by $P_{p_1p_2}$. Similarly, we denote by $\mathcal{P}(p_2)$ the set of all paths containing $P(p_1, p_2)$ and having eccentricity equal to $\frac{\text{diam}}{2} - d(p_2, c)$. Since these two cases are symmetrical, w.l.o.g., we restrict our attention to paths belonging to $\mathcal{P}(p_1)$. We denote the eccentricity of all the paths in $\mathcal{P}(p_1)$ by $E_{\mathcal{P}(p_1)}$.

Among all the paths $P_{p_1p_2} \in \mathcal{P}(p_1)$, we are interested in finding a path which maximizes the minimum distance from all the vertices $u \in T \setminus P_{p_1p_2}$. A path with such a property will be called a *best path of type 2*.

Denote by $c_i, i = 1, 2$ the endvertices of the path center PC. Let (p_1, t) be the edge belonging to $PC \setminus P(p_1, p_2)$ such that t is a child of p_1 in T_{p_1}. Assume that $p_1 \neq c_1, c_2$ and $p_2 \neq c_1, c_2$. In order to find a best path of type 2, we have to find a best path of type 1 starting from p_2 in the subtree T_{p_2} and a best path of type 1 starting from p_1 in the subtree $T_{p_1} \setminus \{T_t \cup (p_1, t)\}$. We denote this latter particular best path of type 1 by $\widehat{P}_{p_1}$. The maximum of the minimum distances of a best path P_{p_2} in T_{p_2} is computed by formula (8), while for $\widehat{P}_{p_1}$ in $T_{p_1} \setminus \{T_t \cup (p_1, t)\}$ it is given by

$$\widehat{\beta}(p_1) = \min\{q\beta(p_1), \ell(p_1, t)\}, \tag{11}$$

where the new label $q\beta(v)$ can be computed during the bottom-up visit of the tree T_c for all the vertices $v \in PC$ and corresponds to formula (8) rewritten by considering the subtree $T_v \setminus \{T_t \cup (v, t)\}$, where $(v, t) \in PC$ and t is a child of v in T_v.

Note that the maximum of the minimum distances of the best paths of type 1 $\widehat{P}_{c_i}, i = 1, 2$, is computed by formula (8) w.r.t. the subtrees $T_{c_i}, i = 1, 2$, respectively. Thus, we set $\widehat{\beta}(c_i) = \beta(c_i), i = 1, 2$.

For a path $P(u_1, u_2)$ we define the function $\mathrm{mil}(P(u_1, u_2))$, i.e., the *minimum incident length*, as the minimum length of an edge not belonging to $P(u_1, u_2)$, but incident to $P(u_1, u_2)$ in one of the vertices g, with $g \neq u_1, u_2$:

$$\mathrm{mil}(P(u_1, u_2)) = \min_{\substack{(f,g)|g \in P(u_1,u_2)\setminus\{u_1,u_2\} \\ f \notin P(u_1,u_2)}} \ell(f, g). \tag{12}$$

Hence, for any given pair of vertices $p_1, p_2 \in PC$ for which a best path of type 2, $P_{p_1p_2}$ belongs to $\mathcal{P}(p_1)$, we denote by $\widehat{M}_{\mathcal{P}(p_1)}(p_2)$ the maximum of the minimum distances from $P_{p_1p_2}$ to all the other vertices of the tree. On the basis of formulas (8), (11) and (12) we have:

$$\widehat{M}_{\mathcal{P}(p_1)}(p_2) = \min\{\widehat{\beta}(p_1), \beta(p_2), \mathrm{mil}(P(p_1, p_2))\}. \tag{13}$$

Now, we turn to consider problems P4–P6 in which, besides minimizing $E(\cdot)$, we want to minimize $\mu(\cdot)$.

Given a rooted tree T_c and a vertex $v \in V$, among all the paths $P_v \in \mathcal{P}(T_v)$ we are interested in finding a path which minimizes the minimum distance from a vertex $u \in T_v \setminus P_v$, to P_v. Also for the Hurwicz-type problems we refer to such a path as a best path of type 1, as long as this does not cause any confusion with a best path of type 1 of the range-type problems.

Since the minimum distance from a vertex to a path is always given by the length of the shortest edge incident to the path, the value of the minimum distance from a vertex $u \in T_v \setminus P_v$, to a best path of type 1, $P_v \in \mathcal{P}(T_v)$, corresponds to the length of the shortest edge contained in T_v that we denote by e_{T_v} (if the shortest edge is not unique, we choose one arbitrarily). More precisely, the following property holds.

Property 3. *Given a rooted tree T_c and a vertex $v \in V$, a best path of type 1 (w.r.t. the Hurwicz problems) $P_v \in \mathcal{P}(T_v)$ is always given by the path in T_v connecting v to the endvertex of e_{T_v} closer to v.* □

After Property 3, in order to compute the minimum of the minimum distances from a vertex $u \in T_v \setminus P_v$, to a best path of type 1 P_v, for each vertex v it suffices to compute the minimum length of an edge in T_v in a bottom-up visit of the tree. We denote this quantity by $h(v)$ and we have:

$$h(v) = \begin{cases} \min\{d_1(v), \min_{w \in S(v)}\{h(w)\}\} & \text{if } |S(v)| \geq 1, \\ +\infty & \text{if } |S(v)| = 0. \end{cases} \tag{14}$$

Then, the minimum of the minimum distances from a best path of type 1, $P_v \in \mathcal{P}(T_v)$, to all the other vertices of the tree is given by:

$$m_{\mathcal{P}(T_v)} = \begin{cases} \min\{\ell(v, p(v)), h(v)\} & \text{if } |S(v)| \geq 1, \\ \ell(v, p(v)) & \text{if } |S(v)| = 0. \end{cases} \tag{15}$$

where $\ell(v, p(v)) = +\infty$ if v is the root of T. When $PC \neq \{c\}$ we evaluate the vertex c as the root of T_2 and, therefore, in formula (15) we set $\ell(c, p(c)) = \ell(c, v_1)$.

We now analyze the paths in $\overline{\mathcal{P}}$ for problems P4–P6 when $PC \neq \{c\}$. As before, we focus our attention only on paths in $\mathcal{P}(p_1)$. Among all the paths $P_{p_1p_2} \in \mathcal{P}(p_1)$ we are interested in finding a path which minimizes the minimum distance from all the vertices $u \in T \setminus P_{p_1p_2}$. A path with such a property will be called a best path of type 2 (as long as this does not cause any confusion with a best path of type 2 of the range-type problems). Assume that $p_1 \neq c_1, c_2$ and $p_2 \neq c_1, c_2$. Let (p_1, t) be the edge belonging to $PC \setminus P(p_1, p_2)$ such that t is a child of $p_1 \in T_{p_1}$. The minimum of the minimum distances of a best path of type 1, $\widehat{P}_{p_1}$ in $T_{p_1} \setminus \{T_t \cup (p_1, t)\}$ is:

$$\widehat{h}(p_1) = \min\{qh(p_1), \ell(p_1, t)\}, \tag{16}$$

where the new label $qh(v)$ can be computed by formula (14) during the bottom-up visit of the tree T_c for all the vertices $v \in PC$, but it has to be rewritten by considering the subtree $T_v \setminus \{T_t \cup (v, t)\}$ where $(v, t) \in PC$ and t is a child of v in T_v. For vertex p_2 we compute the minimum of the minimum distances of a best path of type 1, P_{p_2} in T_{p_2} by formula (14). The minimum of the minimum distances of the best paths of type 1, $\widehat{P}_{c_i}, i = 1, 2$, is computed by formula (14) w.r.t. the subtrees $T_{c_i}, i = 1, 2$, respectively. Thus, we set $\widehat{h}(c_i) = h(c_i), i = 1, 2$.

Table 2
Bottom up labelling procedure

Vertex	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
flag(v)	0	0	0	0	0	0	0	1	0	1	1	0	0	1	0	1	1
$\beta(v)$	∞	14	∞	∞	∞	14	∞	19	∞	9	19	20	∞	4	∞	15	10
$M_{\mathcal{P}(T_v)}$	14	1	17	19	2	12	9	6	20	4	4	7	15	9	33	16	10

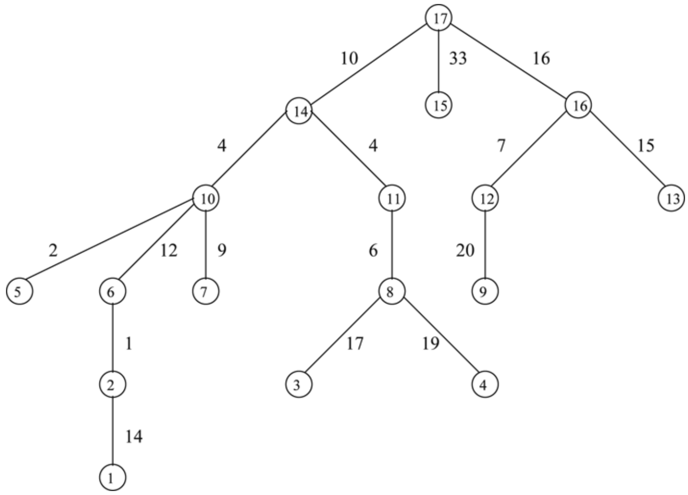

Fig. 3. An example.

Finally, for any given pair of vertices $p_1, p_2 \in PC$ for which a best path of type 2, $P_{p_1p_2}$, belongs to $\mathcal{P}(p_1)$, we denote by $\widehat{m}_{\mathcal{P}(p_1)}(p_2)$ the minimum of the minimum distances from $P_{p_1p_2}$ to all the other vertices of the tree:

$$\widehat{m}_{\mathcal{P}(p_1)}(p_2) = \min\{\widehat{h}(p_1), h(p_2), \mathrm{mil}(P(p_1, p_2))\}. \tag{17}$$

3.1. An example

In this subsection, given a rooted tree T_r (r does not need to be the central vertex), we provide an example showing how formulas (7)–(9) work in order to find the maximum of the minimum distances of a path among all the paths $P_v \in \mathcal{P}(T_v)$, $\forall v \in V$.

In this example we are not considering either the eccentricity of the paths $P_v \in \mathcal{P}(T_v)$ or the partition of the tree T into the two subtrees T_1 and T_2. Here (see Fig. 3), we suppose vertex 17 to be the root of the tree. The values along the edges represent their lengths. For each vertex v, Table 2 reports the values of flag(v), $\beta(v)$, and $M_{\mathcal{P}(T_v)}$.

4. The algorithms

In order to solve problems P1–P6 we adopt a bicriteria approach similar to those presented in [1,15]. Recall that $\pi_i \subseteq \Pi$, $i = 1, 2$, denotes the set of Pareto-optimal paths with respect to the partial order $\succeq_i$, $i = 1, 2$. The algorithms that follow find two supersets $\phi(W_i)$, $i = 1, 2$, that contain the representations of all the Pareto-optimal paths in the outcome space $(E(\cdot), \mu(\cdot))$ with respect to the two partial orders $\succeq_1$ and $\succeq_2$, respectively.

4.1. The Pareto-optimal path representation algorithm for $\succeq_1$

Let us first consider the partial order $\succeq_1$ (i.e., problems P1–P3), and let $\phi(W_1)$ be such that $\phi(\pi_1) \subset \phi(W_1) \subset \phi(\Pi)$, that is, $\phi(W_1)$ contains the representation of all the Pareto-optimal paths w.r.t. $\succeq_1$, along with some extra points. We will show that $\phi(W_1)$ has cardinality $O(n)$.

The idea of the algorithm for computing the set $\phi(W_1)$ is the following: first, the relevant functions are evaluated at all the vertices $v \in V_i$, $i = 1, 2$, and the pairs $(E_{\mathcal{P}(T_v)}, M_{\mathcal{P}(T_v)})$ are included in $\phi(W_1)$. This guarantees that the Pareto-optimal paths belonging to $\mathcal{P}^1$ and $\mathcal{P}^2$ are identified. Then, paths belonging to $\overline{\mathcal{P}}$ are considered, and the pairs $(E_{\mathcal{P}(p_1)}, \widehat{M}_{\mathcal{P}(p_1)}(p_2))$ are added to $\phi(W_1)$. In the latter case, it is not necessary to evaluate all the possible subpaths $P(p_1, p_2) \subseteq PC$, implying an overall time complexity $O(n^2)$. Indeed, it can be shown that it is sufficient to evaluate only a sequence of $O(n)$ subpaths $P_1, \ldots, P_q$ of the path center PC such that, for any given Pareto-optimal path $P \in \overline{\mathcal{P}}$, P is a best path of type 2 for which $P \cap PC \supseteq P_i$ for some $i = 1, 2, \ldots, q$. The following proposition provides a result similar to the one given in [1].

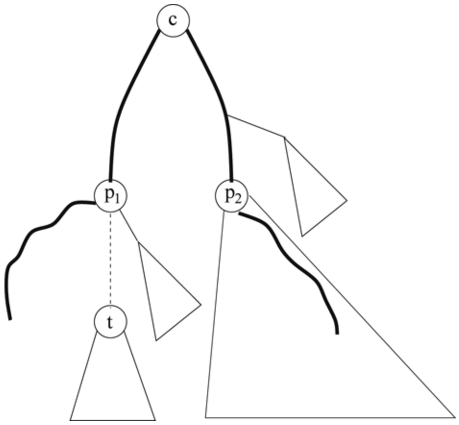

Fig. 4. An Example of a best path of type 2 in $\mathcal{P}(p_1)$.

Proposition 3. *Let $PC = P(c_1, c_2) \neq \{c\}$ and $P(p_1, p_2)$ be a subpath of PC such that $c \in P(p_1, p_2)$, $p_1 \neq c_1$, $p_1 \neq c$, $p_2 \neq c$. Suppose that $\frac{\text{diam}}{2} - d(p_1, c) \geq \frac{\text{diam}}{2} - d(p_2, c)$. Let (p_1, t) be the edge belonging to $PC \setminus P(p_1, p_2)$ such that t is a child of p_1 in T_{p_1}. Consider a path $P \in \overline{\mathcal{P}}$ such that $P \in \pi_1$ and $P(p_1, p_2) \subseteq P \cap PC$. Then, either $t \in P$, or $P = P_{p_1 p_2} \in \mathcal{P}(p_1)$ is a best path of type 2 that satisfies the following two conditions:*

(i) $E(P_{p_1 p_2}) = E_{\mathcal{P}(p_1)} = \frac{\text{diam}}{2} - d(p_1, c)$;
(ii) $\mu(P_{p_1 p_2}) = \widehat{M}_{\mathcal{P}(p_1)}(p_2)$.

Proof. If t is not in P, then, under the assumptions of the proposition, (i) holds, and the path P corresponds to a best path of type 2 in $\mathcal{P}(p_1)$ that can be found only by maximizing $\mu(\cdot)$ through formula (13) (see Fig. 4). $\square$

According to Proposition 3, the Pareto-optimal paths belonging to $\overline{\mathcal{P}}$ can be identified by considering the sequence of subpaths $P_1, \ldots, P_q$. This sequence can be obtained starting from c and adding one edge at a time. Suppose $PC \neq \{c\}$ and recall the decomposition of T presented in Section 2. Let v_1 and v_2 be the two vertices adjacent to c in PC. W.l.o.g., we may refer to v_1 as the vertex such that $\frac{\text{diam}}{2} - d(v_1, c) \geq \frac{\text{diam}}{2} - d(v_2, c)$. Thus, after Proposition 3, we have $P_1 = (c, v_1)$, while the rest of the sequence $P_2, \ldots, P_q$ is generated according to the algorithm for the path center provided in [6] with $P_q = PC$. Actually, we do not consider the absolute center c as a subpath of PC since, when $PC \neq \{c\}$, vertex c alone is evaluated as the root of T_2. Note that the sequence $P_1, \ldots, P_q$ is ordered in non-increasing order w.r.t. the eccentricity, that is, $E(P_1) \geq E(P_2) \geq \cdots \geq E(P_q)$.

The following pseudocode describes the algorithm for finding the set $\phi(W_1)$ with respect to the partial order $\succeq_1$ when $PC \neq \{c\}$. Note that the same algorithm can be adopted when $PC = \{c\}$. In this case Step 4 is skipped, Step 5 must be executed for all $v \in V$, while Step 6 is dropped.

The Pareto-optimal Path Representation Algorithm for $\succeq_1$

Input: An edge-weighted tree T.

Output: The superset $\phi(W_1)$.

1. $\phi(W_1) = \emptyset$
2. Compute the absolute center c and the path center PC. Let v_1 and v_2 be the two vertices adjacent to c in PC such that $\frac{\text{diam}}{2} - d(v_1, c) \geq \frac{\text{diam}}{2} - d(v_2, c)$ and consider $P_1, P_2, \ldots, P_q$, with $P_1 = (c, v_1)$ and $P_q = PC$.
3. Root T at the absolute center c.
4. Identify the two subtrees T_1 and T_2 (see page 7).
5. **For all** $v \in V_i$, $i = 1, 2$ **do**

   $$\phi(W_1) = \phi(W_1) \cup \{(E_{\mathcal{P}(T_v)}, M_{\mathcal{P}(T_v)})\}$$

   endFor
6. **For** $i = 1$ **to** q **do**
   Let $P_i = P(p_1, p_2)$ be the current subpath of PC with $\frac{\text{diam}}{2} - d(p_1, c) \geq \frac{\text{diam}}{2} - d(p_2, c)$. $\phi(W_1) = \phi(W_1) \cup \{(E_{\mathcal{P}(p_1)}, \widehat{M}_{\mathcal{P}(p_1)}(p_2))\}$
   endFor
7. **output** $\phi(W_1)$.

Proposition 4. *The cardinality of the set $\phi(W_1)$ is $O(n)$.*

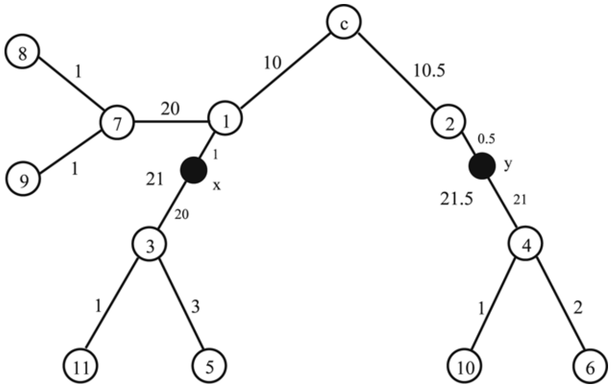

Fig. 5. An optimal solution for the continuous version of problem P1 is given by $P(x, y)$ with $R(P(x, y)) = 23 - 20 = 3$.

Proof. For each vertex $v \in T_i$, with $i = 1, 2$, Theorem 1 and formula (9) uniquely determine the values $(E_{\mathcal{P}(T_v)}, M_{\mathcal{P}(T_v)})$ of a best path in $\mathcal{P}(T_v)$ to be included in $\phi(W_1)$. By Theorem 1, for all the paths $P \in \mathcal{P}(T_v)$, one has $E(P) = E_{\mathcal{P}(T_v)}$, and $\mu(P) \le M_{\mathcal{P}(T_v)}$. Note that the absolute center c is evaluated as a vertex of T_2 in Step 5. Thus, the number of paths considered in the execution of Step 5 is $O(n)$. By formula (10) and formula (13), Step 6 provides the values $(E_{\mathcal{P}(p_1)}, \widehat{M}_{\mathcal{P}(p_1)}(p_2))$ of all the Pareto-optimal paths in $\overline{\mathcal{P}}$. In order to find all such paths it is sufficient to consider only the sequence of $O(n)$ subpaths $P_1, P_2, \dots, P_q$ of PC. Hence, the cardinality of $\phi(W_1)$ is $O(n)$. $\square$

Proposition 5. *The Pareto-optimal Path representation Algorithm for $\succeq_1$ computes the set $\phi(W_1)$ in $O(n)$ time.*

Proof. In the preprocessing phase, labels (6)–(9) and (11) are computed in $O(n)$ time. In Step 2 the absolute center c and the path center PC are computed in time $O(n)$ [6]. In Step 6 we have to compute $E_{\mathcal{P}(p_1)}$ and $\widehat{M}_{\mathcal{P}(p_1)}(p_2)$ for all subpaths in the sequence $P_1, P_2, \dots, P_q$. This requires $O(n)$ time, too. Hence, the overall time complexity of the Pareto-optimal Paths representation Algorithm for $\succeq_1$ is $O(n)$. $\square$

Once the set $\phi(W_1)$ is available, taking into account Propositions 1 and 4, problems P1–P3 can be solved in $O(n)$ time as follows:

Problem P1:

Among all the pairs $(E(P), \mu(P)) \in \phi(W_1)$, find the minimum of $R(P) = E(P) - \mu(P)$.

Problem P2:

For a given $0 \le \gamma \le \max\{\ell(e) | e \in E\}$, find the minimum of $E(P)$ among all the pairs $(E(P), \mu(P)) \in \phi(W_1)$ such that $\mu(P) \ge \gamma$.

Problem P3:

For a given $\gamma \ge 0$, find the maximum of $\mu(P)$ among all the pairs $(E(P), \mu(P)) \in \phi(W_1)$ such that $E(P) \le \gamma$.

In addition, we note that the set $\phi(\pi_1)$ can be extracted from $\phi(W_1)$ in $O(n \log n)$ time by finding the rectilinear lower envelope of the set $\phi(W_1)$ with the algorithm provided by Kapoor [8].

We conclude this section by addressing the case in which the absolute center c is a point along an edge of T. Suppose $(v_1, v_2) \in E$ be the edge containing c. In this case T can be rooted at vertex v_2 and the decomposition described in Section 2 (see page 7) still holds if we consider $T_1 = T_{v_1} = (V_1, E_1)$, and the subtree $T_2 = (V_2, E_2)$ with $V_2 = V \setminus V_1$. Since we are considering only discrete paths, all the above recursive formulas still apply, and the sequence $P_1, P_2, \dots, P_q$ is obtained starting with $P_1 = (v_2, v_1)$.

4.1.1. *The continuous version of the range-type problems*

Unlike the problems in which the optimal location of a path on a tree is found w.r.t. the median criterion, the center criterion, or a convex combination of them, in the case of the range-type problems, it is not true that an optimal solution for the continuous version is always a discrete path (see Fig. 5). Moreover, the analysis applied in the discrete case cannot be extended to find all the nondominated solutions for the continuous versions of problems P1–P3 since there may exist a continuum of such paths. Consider Fig. 5 where the vertices of the tree are denoted by white circles, while the points x and y, located along edges $(1, 3)$ and $(2, 4)$ respectively, are marked in black. All the paths $P(x', y')$ obtained by moving x and y by the same quantity $0 < \epsilon \le 0.5$ towards vertex 3 and 4, respectively, have $E(P(x', y')) = 23 - \epsilon$ and $\mu(P(x', y')) = 20 - \epsilon$, and they are all nondominated.

In the following we provide details about how to solve the continuous versions of problems P1–P3.

First, consider problem P1, i.e., the problem of finding a continuous path P that minimizes $R(P)$. The idea is to show that any optimal continuous path must have its endpoints in a finite set. We can augment the set of vertices V by adding these points, which we call *semi-vertices*, along the edges of T, thus producing an *augmented* set of vertices. Then, the recursive

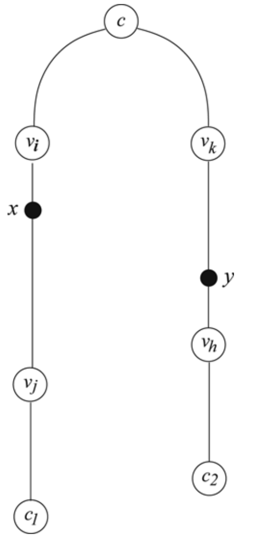

Fig. 6. A continuous path $P(x, y)$ satisfying the assumptions of Lemma 2 such that both (v_i, v_j) and (v_k, v_h) belong to PC.

formulas used in the previous algorithm can be adapted accordingly in order to be applied to the corresponding *augmented* tree rooted at c. For each edge of the tree we add at most $2n$ semi-vertices. Consider edge (v_i, v_j), with $v_i = p(v_j)$, in T_c. For each edge $e \in E$ such that $\ell(e) < \ell(v_i, v_j)$, we add two semi-vertices $x(e)$, $x'(e)$ along (v_i, v_j) such that $d(v_j, x(e)) = \ell(e)$ and $d(v_i, x'(e)) = \ell(e)$. Let SV be the set of all the semi-vertices. Note that the cardinality of SV is $O(n^2)$, and this set can be computed in $O(n^2)$ time by the algorithm provided in [9]. Let $V^a = V \cup SV$. If the center c of the original tree is a point along an edge, then $V^a = V \cup SV \cup \{c\}$. For the sake of simplicity, here we still denote the augmented rooted tree by T_c.

For each edge (v_i, v_j), with $v_i = p(v_j)$, in the original rooted tree, we number the semi-vertices in (v_i, v_j) from the closest to v_j to the closest to v_i, that is, if x_r and x_{r+1} are two *consecutive* semi-vertices in (v_i, v_j), then $d(x_r, v_j) < d(x_{r+1}, v_j)$.

We denote by (v_i, v_j) also the set of all the points along the corresponding edge. Given x_r, x_{r+1} in (v_i, v_j), we denote by (x_{r+1}, x_r) the subset of points of (v_i, v_j) located between x_r and x_{r+1}, that is, for every point $a \in (x_{r+1}, x_r)$ we have $d(x_r, v_j) \leq d(a, v_j) \leq d(x_{r+1}, v_j)$.

The following results show that for solving the continuous version of P1 on the original tree, it is sufficient to consider only paths with endvertices in the finite set V^a.

Lemma 1. *Let (v_i, v_j) and (v_k, v_h) be two edges of the original rooted tree, with $v_i = p(v_j)$ and $v_k = p(v_h)$ (including the case $v_i = v_k$ and $v_j = v_h$). Assume that both (v_i, v_j) and (v_k, v_h) belong to either $T_1 \cup (c, v_1)$ or to T_2. Let $P(x, y)$ be a continuous path with endpoints $x \in (x_{r+1}, x_r)$ in (v_i, v_j) and $y \in (y_{s+1}, y_s)$ in (v_k, v_h) satisfying $d(c, x_{r+1}) < d(c, x_r)$ and $d(c, y_{s+1}) < d(c, y_s)$. Then, there exists a path $P(\hat{x}, \hat{y})$ with $\hat{x}, \hat{y} \in V^a$ such that $R(P(\hat{x}, \hat{y})) \leq R(P(x, y))$.*

Proof. W.l.o.g., suppose $d(x, c) \leq d(y, c)$. For a path $P(x, y)$ satisfying the assumptions of the lemma, only the following two cases hold:

Case 1: $x \in (x_{r+1}, x_r)$ in (v_i, v_j), $x \neq v_i, v_j$, and $y \in (y_{s+1}, y_s)$ in (v_k, v_h), $y \neq v_k, v_h$, and $P(x, y) \subset T_{v_i}$ with $v_i \notin P(x, y)$. We have $E(P(x, y)) = d(x, c) + \frac{\text{diam}}{2}$ and, taking $\hat{x} = x_{r+1}$, $\hat{y} = y_{s+1}$ we get $E(P(\hat{x}, \hat{y})) = E(P(x, y)) - d(x, x_{r+1})$. Then, if $\mu(P(x, y))$ is attained at x, we have $\mu(P(\hat{x}, \hat{y})) = \mu(P(x, y)) - d(x, x_{r+1})$ and, thus, $R(P(\hat{x}, \hat{y})) = R(P(x, y))$. If $\mu(P(x, y))$ is not attained at x it can be verified that $R(P(\hat{x}, \hat{y})) < R(P(x, y))$. In fact, when $\mu(P(x, y))$ is attained at a vertex along $P(x, y)$, we have $R(P(x, y)) = R(P(x, \hat{y})) > R(P(\hat{x}, \hat{y}))$, while, when $\mu(P(x, y))$ is attained at y, we have $R(P(x, y)) > R(P(x, \hat{y})) \geq R(P(\hat{x}, \hat{y}))$.

Case 2: $x \in (x_{r+1}, x_r)$ in (v_i, v_j), $x \neq v_i, v_j$, and $y \in (y_{s+1}, y_s)$ in (v_k, v_h), $y \neq v_k, v_h$, $P(x, y) \subset T_v$, for some v, and $v \in P(x, y)$. In this case we set $\hat{x} = x_{r+1}$, $\hat{y} = y_{s+1}$ and we have $E(P(x, y)) = E(P(\hat{x}, \hat{y}))$, while $\mu(P(\hat{x}, \hat{y})) \geq \mu(P(x, y))$. Hence, $R(P(\hat{x}, \hat{y})) \leq R(P(x, y))$.

It is straightforward to see that the above analysis still holds when one between x and y is a vertex. □

Lemma 2. *Let (v_i, v_j) and (v_k, v_h) be two edges of the original rooted tree, with $v_i = p(v_j)$ and $v_k = p(v_h)$. Assume that $(v_i, v_j) \in T_1 \cup (c, v_1)$ and $(v_k, v_h) \in T_2$. Let $P(x, y)$ be any continuous path with endpoints $x \in (v_i, v_j)$ and $y \in (v_k, v_h)$. There exists a path $P(\hat{x}, \hat{y})$, $\hat{x}, \hat{y} \in V^a$, such that $R(P(\hat{x}, \hat{y})) \leq R(P(x, y))$ for all such continuous paths $P(x, y)$.*

Proof. Let $P(x, y)$ be a continuous path satisfying the assumptions of the lemma. Suppose that both (v_i, v_j) and (v_k, v_h) belong to PC (see, Fig. 6).

In this case, we have

$$E(P(x,y)) = \max\left\{\frac{\text{diam}}{2} - d(c,x), \frac{\text{diam}}{2} - d(c,y)\right\}$$
$$= \frac{\text{diam}}{2} - \min\{d(c,v_j) - d(x,v_j), d(c,v_h) - d(y,v_h)\}, \tag{18}$$

where $d(c,x) = d(c,v_j) - d(x,v_j)$ and $d(c,y) = d(c,v_h) - d(y,v_h)$. On the other hand, the minimum distance is given by

$$\mu(P(x,y)) = \min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\},$$

where $\ell(\bar{e}) = \text{mil}(P(v_i,v_k))$ (see, formula (12)). Note that $\mu(P(x,y)) \le \ell(\bar{e})$. We now compute the range function for $P(x,y)$ as follows

$$R(P(x,y)) = \frac{\text{diam}}{2} - \min\{d(c,v_j) - d(x,v_j), d(c,v_h) - d(y,v_h)\} - \min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\} \tag{19}$$

for which only the following six values are possible:

1. $R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_j) + d(x,v_j) - d(x,v_j) = \frac{\text{diam}}{2} - d(c,v_j)$;
2. $R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_j) + d(x,v_j) - d(y,v_h)$;
3. $R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_j) + d(x,v_j) - \ell(\bar{e})$;
4. $R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_h) + d(y,v_h) - d(y,v_h) = \frac{\text{diam}}{2} - d(c,v_h)$;
5. $R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_h) + d(y,v_h) - d(x,v_j)$;
6. $R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_h) + d(y,v_h) - \ell(\bar{e})$.

For points 1–3, which refer to the cases when the eccentricity of $P(x,y)$ is attained at x, the following holds:

$$R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_j) + \Delta, \quad \Delta \ge 0.$$

In fact, in case 1, $\Delta = 0$. In case 2, we have

$$\Delta = d(x,v_j) - d(y,v_h) \ge 0$$

since $d(y,v_h) = \min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\} \le d(x,v_j)$. Similarly, in case 3, we have

$$\Delta = d(x,v_j) - \ell(\bar{e}) \ge 0$$

since $\ell(\bar{e}) = \min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\} \le d(x,v_j)$.

The same analysis applies to cases 4–6, when the eccentricity of $P(x,y)$ is attained at y, and we have

$$R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_h) + \Delta', \quad \Delta' \ge 0.$$

In any case, the minimum of the range function, say $\bar{R}$, is attained either when $\Delta = 0$, or when $\Delta' = 0$, and, taking into account formula (19), it can be computed as follows:

$$\bar{R} = R(P(x,y)) = \frac{\text{diam}}{2} - \min\{d(c,v_j), d(c,v_h)\}.$$

We show that one can always find two semi-vertices $\hat{x}, \hat{y} \in V^a$ such that $R(P(\hat{x},\hat{y})) = \bar{R}$. Actually, by construction, there always exist two semi-vertices, $x(\bar{e}) \in (v_i,v_j)$ and $y(\bar{e}) \in (v_k,v_h)$ such that $d(x(\bar{e}),v_j) = d(y(\bar{e}),v_h) = \ell(\bar{e})$. Let $\hat{x} = x(\bar{e})$ and $\hat{y} = y(\bar{e})$. Note that it may happen that $\hat{x} = v_i$, or $\hat{y} = v_k$, or both. By formula (19), we have

$$R(P(\hat{x},\hat{y})) = \frac{\text{diam}}{2} - \min\{d(c,v_j) - \ell(\bar{e}), d(c,v_h) - \ell(\bar{e})\} - \ell(\bar{e}) = \bar{R}.$$

Note that there could be other paths $P(x,y)$, with $x \in (v_i,v_j)$ and $y \in (v_k,v_h)$ such that $R(P(x,y)) = \bar{R}$, but all of them are equivalent to $P(\hat{x},\hat{y})$ and, therefore, they can be discarded.

To complete the proof, we consider the cases in which only one between (v_i,v_j) and (v_k,v_h) belongs to PC and the case when none of them belongs to PC.

W.l.o.g, suppose that only (v_i,v_j) belongs to PC. Let v_q be the vertex in PC such that $P(v_i,v_q)$ is the maximum discrete subpath of PC contained in $P(x,y)$. Then

$$R(P(x,y)) = \frac{\text{diam}}{2} - \min\{d(c,v_j) - d(x,v_j), d(c,v_q)\} - \min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\}.$$

When $d(c,v_j) - d(x,v_j) < d(c,v_q)$, the eccentricity of $P(x,y)$ is attained at x, and the analysis is the same as before (see cases 1–3). Otherwise, we have

$$R(P(x,y)) = \frac{\text{diam}}{2} - d(c,v_q) - \min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\} \geq \frac{\text{diam}}{2} - d(c,v_q) - \ell(\bar{e}) = R(P(\hat{x},\hat{y})),$$

since $\min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\} \leq \ell(\bar{e})$.

Finally, when neither (v_i, v_j), nor (v_k, v_h) belong to PC, for any path $P(x,y)$, with $x \in (v_i, v_j)$ and $y \in (v_k, v_h)$, we have $E(P(x,y)) = E(P(v_i, v_k)) = E(P(\hat{x},\hat{y})) = \bar{E}$, and, as before

$$R(P(x,y)) = \bar{E} - \min\{d(x,v_j), d(y,v_h), \ell(\bar{e})\} \geq \bar{E} - \ell(\bar{e}) = R(P(\hat{x},\hat{y})). \quad \square$$

Lemmas 1 and 2 imply that, when searching for an optimal continuous path for problem P1, it is sufficient to consider only those paths with endpoints in V^a. Actually, the continuous range problem on the original tree $T = (V, E)$ is equivalent to the discrete range problem on the augmented tree with the range function defined in the following, more general, way:

$$R(P) = \max_{u \in V' \setminus P} d(u,P) - \min_{u \in V' \setminus P} d(u,P),$$

where $V' \subseteq V^a$ and, in our case, $V' = V$. However, the recursive formulas presented in Section 3 must be suitably adapted. We apply the same decomposition presented for the discrete case (see page 7) to the augmented tree T_c and we still classify paths into *paths of type 1* and *paths of type 2*.

For any $z \in V^a$, let

$$\beta^a(z) = \max_{\substack{P_z \in \mathcal{P}(T_z) \\ z \text{ endvertex of } P_z}} \mu_{T_z}(P_z) \tag{20}$$

i.e., the function $\beta(\cdot)$ is extended to the points in V^a. Consider the edge (v_i, v_j), with $v_i = p(v_j)$, and suppose $z = x_{r+1} \in (v_i, v_j)$. Then, we define

$$\beta^a(x_{r+1}) = \begin{cases} \beta(x_{r+1}) & \text{if } x_{r+1} \in V \\ d(x_{r+1}, v_j) & \text{if } x_{r+1} \in (v_i, v_j), x_{r+1} \neq v_i, v_j \text{ and } |S(v_j)| = 0 \\ \max\{d(x_{r+1}, v_j), \beta^a(x_r)\} & \text{if } x_{r+1} \in (v_i, v_j), x_{r+1} \neq v_i, v_j \text{ and } |S(x_r)| \geq 1, \end{cases} \tag{21}$$

where $x_r \in (v_i, v_j)$, with $d(c, x_{r+1}) < d(c, x_r)$. Note that both x_{r+1} and x_r may be either original vertices (that is, $x_{r+1} = v_i$, $x_r = v_j$) or semi-vertices.

In formula (21) we can set $\beta^a(x_{r+1}) = \beta(x_{r+1})$ when x_{r+1} is an original vertex, since it is easy to check that, in the augmented tree, for any best path of type 1, $P(v_i, y) \in \mathcal{P}(T_{v_i})$, with one end at vertex v_i and the other end at a semi-vertex $y \in (v_k, v_h)$, the path $P(v_i, v_k) \subset P(v_i, y)$ is a best path of type 1, with both its endpoints at original vertices, for which $R(P(v_i, v_k)) \leq R(P(v_i, y))$.

For a given $z \in V^a$, we define the function $M^a_{\mathcal{P}(T_z)}$ as follows:

$$M^a_{\mathcal{P}(T_z)} = \begin{cases} M_{\mathcal{P}(T_z)} & \text{if } z \in V \\ \min\{d(z, v_i), \beta^a(z)\} & \text{if } z \in (v_i, v_j), z \neq v_i, v_j. \end{cases} \tag{22}$$

Now consider paths of type 2. Let PC be the path center in the augmented tree T_c. Consider any path $P(z_1, z_2) \subseteq PC$ with $z_1, z_2 \in V^a$. We always refer to z_1 as the vertex at which the eccentricity of $P(z_1, z_2)$ is attained, and we denote by $\mathcal{P}(z_1)$ the set of all paths of type 2 in the augmented tree containing $P(z_1, z_2)$ and having eccentricity equal to $\frac{\text{diam}}{2} - d(z_1, c)$. In order to find a best path of type 2 in $\mathcal{P}(z_1)$, for all $z \in PC$ we compute:

$$\widehat{\beta}^a(z) = \begin{cases} \widehat{\beta}(z) & \text{if } z \in V \\ d(z, v_j) & \text{if } z \in (v_i, v_j), z \neq v_i, v_j. \end{cases} \tag{23}$$

Finally, we have:

$$\widehat{M}^a_{\mathcal{P}(z_1)}(z_2) = \min\{\widehat{\beta}^a(z_1), \beta^a(z_2), \text{mil}(P(z_1, z_2))\}. \tag{24}$$

Note that, when an original vertex $z \in V$ is considered, the recursive formulas $\beta(z)$, $M_{\mathcal{P}(T_z)}$, and $\widehat{\beta}(z)$, in (21)–(23), respectively, are computed taking into account that a child w of z could be either an original vertex or a semi-vertex.

These formulas can be computed on the augmented tree in a preprocessing phase in time $O(|V^a|)$. Hence, the algorithm for the discrete case can be applied to solve the continuous version of problem P1 with an overall time complexity $O(n^2)$.

Now, consider problem P2, i.e., the problem of finding a continuous path P that minimizes $E(P)$ with $\mu(P) \geq \gamma$, for a given γ, $0 \leq \gamma \leq \max\{\ell(e) | e \in E\}$. We root the tree at c, and augment the set of its vertices V by adding to each edge $e = (v_i, v_j)$ with $v_i = p(v_j)$ and $\ell(e) > \gamma$, two new semi-vertices $x(\gamma)$ and $x'(\gamma)$ such that $d(v_j, x(\gamma)) = \gamma$ and $d(v_i, x'(\gamma)) = \gamma$. Let V^γ denote the augmented set of vertices. Note that at most two new semi-vertices are added for each edge and thus we have $|V^\gamma| = O(n)$. For the sake of simplicity, we still denote the rooted augmented tree by T_c.

Proposition 6. *Let (v_i, v_j) and (v_k, v_h) be two edges of the original rooted tree, with $v_i = p(v_j)$ and $v_k = p(v_h)$ (including the case $v_i = v_k$ and $v_j = v_h$). Consider problem* P2 *for a given γ, $0 \leq \gamma \leq \max\{\ell(e) | e \in E\}$. Let $P(x, y)$ be any feasible continuous path with endpoints $x \in (v_i, v_j)$ and $y \in (v_k, v_h)$. There exists a feasible path $P(\hat{x}, \hat{y})$, $\hat{x}, \hat{y} \in V^\gamma$, such that $E(P(\hat{x}, \hat{y})) \leq E(P(x, y))$ for all such feasible continuous paths $P(x, y)$.*

Proof. Let $P(x, y)$ be a feasible continuous path satisfying the assumptions of the proposition. Let V^γ be the augmented set of vertices and $x(\gamma), y(\gamma) \in V^\gamma$ the two semi-vertices such that $x(\gamma) \in (v_i, v_j)$, with $d(x(\gamma), v_j) = \gamma$, and $y(\gamma) \in (v_k, v_h)$, with $d(y(\gamma), v_h) = \gamma$. First, suppose that the vertices v_i and v_k are contained in $P(x, y)$, v_j and v_h are not, and $x \neq v_i, v_j, x(\gamma)$, $y \neq v_k, v_h, y(\gamma)$. Since $P(x, y)$ is feasible, we have $\mu(P(x, y)) \geq \gamma$, $d(x, v_j) > \gamma$ and $d(y, v_h) > \gamma$. Hence, setting $\hat{x} = x(\gamma)$ and $\hat{y} = y(\gamma)$ produces a new discrete path (in the augmented tree) $P(\hat{x}, \hat{y})$ such that $\mu(P(\hat{x}, \hat{y})) = \gamma$. If the eccentricity of $P(x, y)$ is not attained at x nor at y, then $E(P(\hat{x}, \hat{y})) = E(P(x, y))$, otherwise $E(P(\hat{x}, \hat{y})) < E(P(x, y))$.

In the particular case when x and y belong to the same edge, that is, $v_i = v_k$ and $v_j = v_h$, w.l.o.g., we can assume $d(x, c) < d(y, c)$, and we set $\hat{x} = x'(\gamma)$, where $d(v_i, x'(\gamma)) = \gamma$, and $\hat{y} = y(\gamma)$.

A similar analysis applies to all the other possible configurations of $P(x, y)$. □

After Proposition 6, we are able to compute analogous formulas to (20)–(24) for the augmented tree w.r.t. the set V^γ and solve the problem of finding a discrete path P that minimizes $E(P) = \max_{u \in V' \setminus P} d(u, P)$ with $\mu(P) = \min_{u \in V' \setminus P} d(u, P) \geq \gamma$ in the augmented tree, where $V' \subseteq V^\gamma$ and, in our case, $V' = V$. This problem is equivalent to the continuous version of problem P2 on the original tree. The new formulas can be computed on the augmented tree in a preprocessing phase in time $O(|V^\gamma|)$. Hence, an optimal solution for the continuous version of problem P2 can be obtained in $O(n)$ time.

Now consider problem P3, i.e., the problem of finding a continuous path P that maximizes $\mu(P)$ with $E(P) \leq \gamma$, for a given $\gamma \geq 0$. Recall that $E(PC) \leq \gamma$ must hold, otherwise the problem is infeasible.

Even in this case we root the tree at c, and augment the set of its vertices by adding $O(n)$ new semi-vertices to V. The following proposition provides a result for the unconstrained continuous version of P3, i.e., finding a continuous path P that maximizes $\mu(P)$.

Proposition 7. *A continuous path that maximizes the minimum distance $\mu(\cdot)$ in a tree T_c is either discrete or it is the middle point of the longest edge of T_c.*

Proof. Suppose that $P(x, y)$ is a continuous path that maximizes the minimum distance $\mu(\cdot)$. If x and y are points belonging to the same edge, then $P(x, y)$ must be a single point (i.e., $x = y$), and it must coincide with the middle point of the edge. Moreover, it is easy to check that, since $P(x, y)$ is optimal, this case occurs only when $x = y$ is the middle point of the longest edge of T_c. On the other hand, suppose x and y belong to different edges and at least one of them is not a vertex. W.l.o.g., we can always assume that x is a vertex and y is in the interior of an edge (v_i, v_j), with $v_i = p(v_j)$, and that v_i belongs to $P(x, y)$, but v_j does not. Then, for the (unique) discrete path with the same set of vertices of $P(x, y)$, $P(x, v_i) \subset P(x, y)$, $\mu(P(x, v_i)) = \mu(P(x, y))$ must hold, otherwise $P(x, y)$ cannot be optimal. By similar arguments, it can be shown that if $P(x, y)$ is a continuous path that maximizes the minimum distance with both x and y points along some edges, then, even in this case, the (unique) discrete path with the same set of vertices of $P(x, y)$, and contained in $P(x, y)$, is optimal as well. □

Proposition 7 shows that the middle point of each edge is a possible candidate for the optimal solution of the continuous version of problem P3. Thus, we augment V by adding new semi-vertices corresponding to the middle points of all the edges. Let MP be the set of all the middle points, then, we augment V to $V \cup MP$.

For the sake of simplicity, we still denote the rooted augmented tree by T_c. We consider the usual decomposition of T_c (see page 7) and refer to the classification of paths in *paths of type 1* and *paths of type 2*.

However, since in P3 the constraint on the eccentricity must be satisfied, some more semi-vertices must be added. For finding the best paths of type 1, we further augment V and, along each path from c to a leaf v of T_c for which $d(c, v) \geq \gamma - \frac{\text{diam}}{2}$, we add a new vertex $x(\gamma)$ such that $d(x(\gamma), c) + \frac{\text{diam}}{2} = \gamma$. Let (u, v), with $v = p(u)$ and $u, v \in V \cup MP$, be an edge that contains an additional vertex $x(\gamma)$. Since, for any point z in $T_{x(\gamma)} \setminus \{x(\gamma)\}$, $d(z, c) + \frac{\text{diam}}{2} > \gamma$, all the paths of type 1 in $T_{x(\gamma)} \setminus \{x(\gamma)\}$ are infeasible.

For finding the best paths of type 2, we only include two additional vertices, $x_1(\gamma)$ and $x_2(\gamma)$, along the path center such that $\gamma = \frac{\text{diam}}{2} - d(c, x_1(\gamma)) = \frac{\text{diam}}{2} - d(c, x_2(\gamma))$. Let (v_1, u_1) and (v_2, u_2), with $v_1 = p(u_1)$ and with $v_2 = p(u_2)$, be the two edges of PC containing $x_1(\gamma)$ and $x_2(\gamma)$, respectively. Note that all the paths of type 2 that do not contain both $x_1(\gamma)$ and $x_2(\gamma)$ are infeasible.

We denote the set of all these additional vertices by Γ. Thus, for problem P3 we further augment $V \cup MP$ to $V^\gamma = V \cup MP \cup \Gamma$, with $|V^\gamma| = O(n)$, and this set can be computed in linear time.

For a given $\widehat{x} \in \Gamma$, $\widehat{x} \neq x_1(\gamma), x_2(\gamma)$, let (v_i, v_j) be the edge of the original tree that contains $\widehat{x}$, with $v_i = p(v_j)$, and let $u \in MP$ be the middle point of (v_i, v_j). W.l.o.g., assume $\widehat{x} \in (v_i, u)$, $\widehat{x} \neq v_i, u$. Then, in the augmented tree, $P_{\widehat{x}}$ is a feasible path of type 1. Any continuous feasible path $P(z)$ that can be obtained by extending $P_{\widehat{x}}$ up to a point $z \in (v_i, \widehat{x})$, $z \neq v_i, \widehat{x}$, can be discarded since we always have $\mu(P_{\widehat{x}}) \geq \mu(P(z))$.

On the other hand, consider $x_1(\gamma)$ and $x_2(\gamma)$ and every feasible continuous path $P(z_1, z_2)$ for which z_1 and z_2 are not vertices in V^γ and $P(z_1, z_2) \cap PC \supseteq P(x_1(\gamma), x_2(\gamma))$. Let u_i,u_j,u_k,u_h be vertices in the corresponding augmented tree such that $u_i = p(u_j)$, with $z_1 \in (u_i, u_j)$, $z_1 \neq u_i, u_j$, and $u_k = p(u_h)$, with $z_2 \in (u_k, u_h)$, $z_2 \neq u_k, u_h$. Then, the path $P(u_i, u_k)$ is feasible and such that $P(u_i, u_k) \cap PC \supseteq P(x_1(\gamma), x_2(\gamma))$, and we have $\mu(P(u_i, u_k)) \geq \mu(P(z_1, z_2))$. Hence, even these (continuous) paths $P(z_1, z_2)$ can be discarded.

The above discussion guarantees that, for solving the continuous version of problem P3, it is sufficient to consider only those paths with endpoints in V^γ. Actually, we are able to compute analogous formulas to (20)–(24) for the augmented tree w.r.t. the set V^γ and solve the problem of finding a discrete path P maximizing $\mu(P) = \min_{u \in V' \setminus P} d(u, P)$ with $E(P) =$

$\max_{u\in V'\setminus P} d(u, P) \leq \gamma$ in the augmented tree, where $V' \subseteq V^\gamma$ and, in our case, $V' = V$. This problem is equivalent to the continuous version of problem P3 on the original tree. The new formulas can be computed on the augmented tree in a preprocessing phase in time $O(|V^\gamma|)$. Hence, an optimal solution for the continuous version of problem P3 can be obtained in $O(n)$ time.

4.2. The Pareto-optimal path representation algorithm for $\succeq_2$

In this section we consider the partial order $\succeq_2$, i.e., problems P4–P6. Let $\phi(W_2)$ be such that $\phi(\pi_2) \subset \phi(W_2) \subset \phi(\Pi)$, that is, $\phi(W_2)$ contains the representation of all the Pareto-optimal paths w.r.t. $\succeq_2$, along with some extra points. We will show that $\phi(W_2)$ has cardinality $O(n)$.

The idea of the algorithm for computing the set $\phi(W_2)$ is the following: first, the relevant functions are evaluated at all the vertices $v \in V_i$, $i = 1, 2$, and the pairs $(E_{\mathcal{P}(T_v)}, m_{\mathcal{P}(T_v)})$ are included in $\phi(W_2)$. This guarantees that the Pareto-optimal paths belonging to $\mathcal{P}^1$ and $\mathcal{P}^2$ are identified. Then, paths belonging to $\overline{\mathcal{P}}$ are considered, and the pairs $(E_{\mathcal{P}(p_1)}, \widehat{m}_{\mathcal{P}(p_1)}(p_2))$ are added to $\phi(W_2)$. The following Proposition 8 provides results similar to those of Proposition 3, with respect to a suitable sequence of $O(n)$ subpaths $P_1, \ldots, P_q$ of PC.

Proposition 8. *Let $PC = P(c_1, c_2) \neq \{c\}$ and $P(p_1, p_2)$ be a subpath of PC such that $c \in P(p_1, p_2)$, $p_1 \neq c_1$, $p_1 \neq c$, $p_2 \neq c$. Suppose that $\frac{\text{diam}}{2} - d(p_1, c) \geq \frac{\text{diam}}{2} - d(p_2, c)$. Let (p_1, t) be the edge belonging to $PC \setminus P(p_1, p_2)$ such that t is a child of p_1 in T_{p_1}. Consider a path $P \in \overline{\mathcal{P}}$ such that $P \in \pi_2$ and $P(p_1, p_2) \subseteq P \cap PC$. Then, either $t \in P$, or $P = P_{p_1p_2} \in \mathcal{P}(p_1)$ is a best path of type 2 that satisfies the following two conditions:*

(i) $E(P_{p_1p_2}) = E_{\mathcal{P}(p_1)} = \frac{\text{diam}}{2} - d(p_1, c)$;
(ii) $\mu(P_{p_1p_2}) = \widehat{m}_{\mathcal{P}(p_1)}(p_2)$.

Proof. If t is not in P, then, under the assumptions of the proposition, (i) holds, and path P corresponds to a best path of type 2 in $\mathcal{P}(p_1)$ that can be found only by minimizing $\mu(\cdot)$ through formula (17) (see Fig. 4). □

According to Proposition 8, the Pareto-optimal paths belonging to $\overline{\mathcal{P}}$ can be identified by considering the sequence of subpaths $P_1, \ldots, P_q$. This sequence can be obtained starting from c and adding one edge at a time. Suppose $PC \neq \{c\}$ and recall the decomposition of T presented in Section 2. Let v_1 and v_2 be the two vertices adjacent to c in PC. W.l.o.g., we may refer to v_1 as the vertex such that $\frac{\text{diam}}{2} - d(v_1, c) \geq \frac{\text{diam}}{2} - d(v_2, c)$. Thus, after Proposition 8, we have $P_1 = (c, v_1)$, while the rest of the sequence $P_2, \ldots, P_q$ is generated according to the algorithm for the path center provided in [6] with $P_q = PC$. Actually, we do not consider the absolute center c as a subpath of PC since, when $PC \neq \{c\}$, vertex c alone is evaluated as the root of T_2.

In order to obtain the Pareto-optimal Path Representation Algorithm for $\succeq_2$, the pseudocode provided in Section 4.1 can be re-arranged by using formulas (15) and (17) in place of (9) and (13), respectively. The case in which the absolute center c is a point along an edge of T is handled as in Section 4.1.

Proposition 9. *The cardinality of the set $\phi(W_2)$ is $O(n)$.*

Proof. On the basis of Proposition 8, the proof uses arguments similar to those provided in the proof of Proposition 4. □

Proposition 10. *The Pareto-optimal Path representation Algorithm for $\succeq_2$ computes the set $\phi(W_2)$ in $O(n)$ time.*

Proof. In the preprocessing phase, labels (6) and (14)–(16) are computed in $O(n)$ time. The absolute center c and the path center PC are computed in time $O(n)$ [6]. The computation of $E_{\mathcal{P}(p_1)}$ and $\widehat{m}_{\mathcal{P}(p_1)}(p_2)$ for all the subpaths of the sequence $P_1, P_2, \ldots, P_q$ requires $O(n)$ time. Hence, the overall time complexity of the algorithm is $O(n)$. □

Once the set $\phi(W_2)$ is available, we are able to solve problems P4–P6. In addition, we note that the set $\phi(\pi_2)$ can be extracted from $\phi(W_2)$ in time $O(n \log n)$ by finding the rectilinear lower envelope of the set $\phi(W_2)$ with the algorithm provided by Kapoor [8].

Taking into account Propositions 1 and 9, problems P4–P6 can be solved in $O(n)$ time as follows:

Problem P4:
Given $0 \leq \alpha \leq 1$, among all the pairs $(E(P), \mu(P)) \in \phi(W_2)$, find the minimum of $H(P) = \alpha E(P) + (1 - \alpha)\mu(P)$.

Problem P5:
For a given $\gamma \geq \min\{\ell(e)|e \in E\}$, find the minimum of $E(P)$ among all the pairs $(E(P), \mu(P)) \in \phi(W_2)$ such that $\mu(P) \leq \gamma$.

Problem P6:
For a given $\gamma \geq 0$, find the minimum of $\mu(P)$ among all the pairs $(E(P), \mu(P)) \in \phi(W_2)$ such that $E(P) \leq \gamma$.

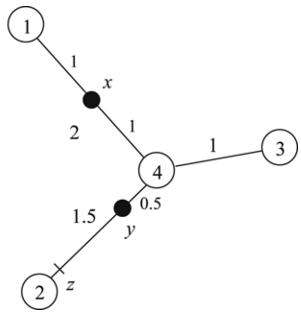

Fig. 7. The path center is $P(x, y)$, where x and y are points in the interior of edges $(1, 4)$ and $(2, 4)$, respectively, and $E(P(x, y)) = 1$. Moving y up to z produces the path $P^{\varepsilon} = P(x, z)$ that has the same eccentricity as $P(x, y)$ and minimum distance equal to ε. However, taking for instance $\alpha = 0.5$, we have $H(P^{\varepsilon}) = 0.5 + 0.5\varepsilon$, but when $\varepsilon = 0$, that is, z coincides with vertex 2, we have $H(P^0) = 1$. Thus, $\lim_{\varepsilon\to 0} H(P^{\varepsilon}) \neq H(P^0)$, implying that the value 0.5 cannot be reached.

4.2.1. The continuous version of the Hurwicz-type problems

The results for the continuous versions of problems P4–P6 rely on the following proposition.

Proposition 11. *Given a tree T that is not a path, for all $\varepsilon > 0$, one can always find a path P such that $E(P) = E(PC)$ and $\mu(P) = \varepsilon$.*

Proof. First of all, note that in the continuous case the path center PC will never have its endpoints in the leaves (unless the tree is a path), since, according to the definition of path center, the minimum length path that minimizes the maximum distance from the vertices of the tree will always end before reaching a leaf, at a distance equal to the eccentricity from that leaf. Thus, no endpoint of PC is a leaf (see, for example, the path $P(x, y)$ in Fig. 7) and, for every $\varepsilon > 0$, a path P^{ε} such that $E(P^{\varepsilon}) = E(PC)$ and $\mu(P^{\varepsilon}) = \varepsilon$ can be obtained by enlarging PC along an edge, from one of its endpoints up to a distance equal to ε from the next vertex (see the path $P(x, z)$ in Fig. 7). □

Proposition 11 shows that there exists no optimal solution for problems P4 and P6, since, in both cases, for any feasible path $P^{\bar{\varepsilon}}$, a better feasible solution P^{ε} can always be found with $0 < \varepsilon < \bar{\varepsilon}$, but the infimum of the objective function cannot be reached. This situation is shown in Fig. 7 for problem P4 with the value $\alpha = 0.5$. Moreover, Proposition 11 shows that problem P5 is feasible for every $\gamma > 0$, and an optimal solution can be always obtained by P^{ε}, with $\varepsilon \leq \gamma$. In this case problem P5 reduces to computing the path center. On the other hand, if $\gamma = 0$ problem P5 is infeasible.

5. Concluding remarks

In this paper we study the problem of locating a path on a network with different objective functions conceptually related to the variability of the distribution of the distances from the demand points to the path. We formulate six different problems (Problems P1–P6), where the first three problems are related to the range objective function and the other three to the Hurwicz objective function. We show that all the considered problems are NP-hard on general networks.

We provide a dynamic programming approach to solve the discrete version of all the problems on trees in $O(n)$ time. In addition, we define two partial orders induced by the maximum and the minimum distance criteria, and show that, for the discrete problems on a tree, a representation of the set of all the Pareto-optimal paths, with respect to these partial orders, can be obtained in $O(n \log n)$ time.

We also discuss the continuous versions of the range-type and Hurwicz-type problems on trees. For Problem P1, that is, finding a continuous path that minimizes the range function, we provide a $O(n^2)$ time algorithm for finding an optimal solution, while for problems P2 and P3 we provide linear time algorithms. For the continuous version of the Hurwicz-type problems we show that either an optimal path does not exist (Problems P4 and P6), or it can be found in constant time once the path center is available (Problem P5).

We note that, since in the discrete case our algorithms are able to generate the whole Pareto-optimal path representation set, they can also be used to solve the following, more general, problem: find a discrete path P which minimizes the linear combination of $E(P)$ and $\mu(P)$: $\lambda E(P) + \delta\mu(P)$, where $\lambda \geq 0$ and $\delta \in \mathbb{R}$.

It is still an open problem how to extend the algorithms presented in this paper to the case in which nonnegative weights are assigned to the vertices of the tree. According to similar results in [15], we conjecture that subquadratic time algorithms exist for most of the weighted versions of these problems. Analyzing these cases will be the subject of a follow up paper.

Acknowledgements

The authors want to thank Prof. Arie Tamir for his contribution to better understanding problems P2 and P3 in the continuous case. The research of the first author is partially supported by Spanish research grant numbers: MTM2007-67433-C02-01, P06-FQM-01366 and HI2006-0123. The research of the second author is supported by the Italian grant number “Azioni Integrate Italia-Spagna” 2.1.4.4.1.15 and Spanish research grant number MTM2007-67433-C02-01. The research of the third author is partially supported by the Italian research grants “Azioni Integrate Italia-Spagna” (number 2.1.4.4.1.15).

References

[1] I. Averbakh, O. Berman, Algorithms for path medi-centers of a tree, Computer and Operations Research 26 (1999) 1395–1409.
[2] R.I. Becker, I. Lari, A. Scozzari, Algorithms for central-median paths with bounded length on trees, European Journal of Operational Research 179 (2007) 1208–1220.
[3] B. Boffey, J.A. Mesa, A review of extensive facility location in networks, European Journal of Operational Research 95 (1996) 592–600.
[4] T. Cáceres, M.C. López-de-los-Mozos, J.A. Mesa, The path-variance problem on tree networks, Discrete Applied Mathematics 145 (2004) 72–79.
[5] S.L. Hakimi, E.F. Schmeichel, M. Labbé, On locating path- or tree- shaped facilities on networks, Networks 23 (1993) 543–555.
[6] S.M. Hedetniemi, E.J. Cockayne, S.T. Hedetniemi, Linear algorithms for finding the Jordan center and path center of a tree, Transportation Science 15 (1981) 98–114.
[7] L. Hurwicz, Optimality criteria for decison making under ignorance. Cowles Comission Discussion Paper No. 370, 1951.
[8] S. Kapoor, Dynamic maintenance of maxima of 2-d point sets, SIAM Journal of Computing 29 (2000) 1858–1877.
[9] T.U. Kim, T.J. Lowe, A. Tamir, J.E. Ward, On the location of a tree-shaped facility, Networks 28 (1996) 167–175.
[10] M.T. Marsh, D.A. Schilling, Equity measurement in facility location analysis: A review and framework, European Journal of Operational Research 74 (1994) 1–17.
[11] J.A. Mesa, J. Puerto, A. Tamir, Improved algorithms for several network location problems with equality measures, Discrete Applied Mathematics 130 (2003) 437–448.
[12] J. Milnor, Games against nature, in: R. Thrall, C. Coombs, R. Davis (Eds.), Decision Processes, John Wiley, 1954, pp. 49–59.
[13] D. Pérez-Brito, J.A. Moreno-Pérez, The generalized p-centdian on networks, TOP 8 (2000) 265–286.
[14] J. Puerto, F. Ricca, A. Scozzari, The continuous and discrete path-variance problems on trees. Networks (in press).
[15] J. Puerto, A.M. Rodríguez-Chía, A. Tamir, D. Pérez-Brito, The bicriteria double weighted median-center path problem on a tree, Networks 47 (2006) 237–247.
[16] J. Puerto, A. Tamir, Locating tree-shaped facilities using the ordered median objective, Mathematical Programming A 102 (2005) 313–338.
[17] A. Shioura, M. Shigeno, The tree center problems and the relationship with the bottleneck knapsack problems, Networks 29 (1997) 107–110.
[18] P.J. Slater, Locating central paths in a graph, Transportation Science 16 (1982) 1–18.
[19] A. Tamir, Fully polynomial approximation schemes for locating a tree-shaped facility: A generalization of the knapsack problem, Discrete Applied Mathematics 87 (1998) 229–243.
[20] A. Tamir, J. Puerto, D. Pérez-Brito, The centdian subtree on tree networks, Discrete Applied Mathematics 118 (2002) 263–278.
[21] A. Tamir, J. Puerto, J.A. Mesa, A.M. Rodríguez-Chía, Conditional location of path and tree shaped facilities on trees, Journal of Algorithms 56 (2005) 50–75.